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Inefficiency of Nash Equilibria in Strategic Market Games

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January, 1985

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P. Dubey's research was supported, in part, by an NSF grant.

ABSTRACT

We examine the efficiency properties of an abstractly given market mechanism. This consists of a smooth map from traders' strategy-choices to their net trades. When the number of traders is finite (the oligopolistic case), it is shown that Nash equilibria "tend to be" inefficient for "most" utilities. The phenomenon is analyzed via a certain set of "ultra-optimal" points that are determined solely by the mechanism. In the last section we apply our results to the Shapley-Shubik mechanisms by way of an illustration.

Inefficiency of Nash Equilibria in Strategic Market Games

by

P. Dubey and J. D. Rogawski

1. Introduction

In this paper we examine the efficiency properties of an abstractly given market mechanism. This consists of a map from traders' strategy choices to their net trades. We restrict attention to the case for which the map is smooth. The classical example is, of course, Cournot's partial equilibrium model of 1838. In the last few years there has been a revival of interest in Cournot's basic model in a general equilibrium framework (see, in particular, the articles and references in [5]). These analyses center on the Nash Equilibria (N.E.) of strategic games induced by market mechanisms. A key question that naturally arises is: to what extent do the N.E. yield efficient allocations? The issue was discussed in [4] for a continuum of traders. It was shown in [4] that if, in addition to being smooth, a mechanism satisfies certain other properties, then its N.E. allocations are always efficient--indeed they are Walrasian. Our paper may be regarded as dual to [4], for its overall result is: when there is a finite number of traders, the N.E. "tend to be" generically inefficient. Two special instances of this result have been noted earlier. In [2] the inefficiency was established under the condition that the dimension of each trader's strategy set is at most $\ell-1$, where ℓ is the number of commodities in the economy. This condition was dropped in [4], but a much weaker assertion was made, namely: for any mechanism there exists an open set of economies, each of which has at least one inefficient N.E.

Expressing the hope that it should be possible to go much further, Mas Colell observed ([6]) that "the issue is more complex than it appears at first sight. Indeed, the incentive compatibility literature (see the recent Review of Economic Studies [7] Symposium) has taught us that it is possible to design mechanisms yielding Pareto Optimal [i.e., efficient] noncooperative [i.e., Nash] equilibria." We sketch a proof (see remark 3.4 in section 3) that a smooth mechanism with efficient N.E. is exceptional; in other words, if such a mechanism is perturbed a little, inefficiency will reappear. More precisely, for a generic choice of mechanisms and of traders' utilities, efficient N.E. are submanifolds of positive codimension within the N.E. manifold.

But, if one is interested in any particular mechanism, it is impossible to decide whether it is "generic" or not and the above result tells us nothing. The bulk of our paper concerns an arbitrarily fixed mechanism. Thus specific examples can be studied by our methods. In section 4, we analyze the "sell-all" and "buy-sell" models of Shapley and Shubik by way of an illustration.

Let us briefly outline our results. Consider an exchange economy with ℓ commodities and n traders. Denote by $a_j \in R_+^\ell$ the initial endowment of trader j and by u_j: $R_+^\ell \to R$ his utility function. Further suppose that a smooth manifold S_j is given for each j as his strategy set. Let $S = S_1 x \ldots x S_n$. A market mechanism $\Phi = (\Phi_1, \ldots, \Phi_n)$ consists of n smooth maps

$$\Phi_j: \ S \to R_+^\ell ,$$

where $\sum_{j=1}^{n} \Phi_j(\bar{s}) = \sum_{j=1}^{n} a_j$ for all $\bar{s} \in S$. Thus Φ defines reallocations of the initial endowment as a function of traders' strategy-choices. For any $\bar{s} = (s_1, \ldots, s_n)$, let $Y_j(\bar{s}) = \{\Phi_j(s_1, \ldots, s_{j-1}, t, s_{j+1}, \ldots, s_n):$ $t \in S_j\}$ be the set of commodity bundles that j can obtain by changing his own strategy while the other players remain fixed according to $\bar{s}$. We will assume, given our smooth context, that $Y_j(\bar{s})$ is a submanifold of R_+^{ℓ}. (In the terminology of [4], this ensures that all the N.E. are "proper.")

If the dimension of $Y_j(\bar{s})$ were ℓ, trader j could move to the "northeast" of R_+^{ℓ} (thereby increasing his holding of each commodity) and no N.E. would exist. It is natural to rule this out and to require that he cannot get "something from nothing." We take $\dim Y_j(\bar{s}) \leq \ell-1$ from now on.

The analysis is broken into two cases. First suppose that $\dim Y_j(\bar{s}) = k_j \leq \ell-1$ for all $1 \leq j \leq n$ and $\bar{s} \in S$, with $k_j < \ell-1$ for at least one j. Put $k = k_1 + \ldots + k_n$ and $k_* = \max_{1 \leq j \leq n} k_j$. Also denote by $N(\bar{u}) \subset S$ the set of N.E. when utilities $\bar{u} = (u_1, \ldots, u_n)$ are assigned to the traders, and by $EN(\bar{u}) \subset N(\bar{u})$ the set of efficient N.E. We show (remark 3.3) that, for a generic choice of $\bar{u}$, $EN(\bar{u})$ has codimension at leat $k_* + (n-1)(\ell-1) - k > 0$ in $N(\bar{u})$. In short "most" N.E.'s are inefficient for generic utilities.

It remains to consider the case when $\dim Y_j(\bar{s}) = \ell-1$ for all $1 \leq j \leq n$ and $\bar{s} \in S$. This is perhaps the most natural from an economic point of view (and was, incidentally, made an assumption in [4], see page 235). For the interpretation, suppose that trading is done via prices. (Our set-up does not need this restriction but, of course,

does not exclude it either). Then a strategy choice $\bar{s} \in S$ produces not only the allocation $\Phi_1(\bar{s}), \ldots, \Phi_n(\bar{s})$, but also prices $p(\bar{s}) \in R^{\ell}_{+}$. The net trade $\Phi_j(\bar{s}) - a_j$ of any j must satisfy the "budget-constraint" $p(\bar{s})\cdot(\Phi_j(\bar{s}) - a_j) = 0$, where '$\cdot$' denotes dot product. In this situation the assumption $\dim Y_j(\bar{s}) = \ell-1$ means that j can enter freely as a buyer or a seller for each commodity, provided only that he balance his budget. If prices were constant we would get the familiar "budget hyperplane" of Walrasian analysis. Here we have, more generally, a hypersurface because each trader affects the prices by his actions, which is of the essence in such games.

When $\dim Y_j(\bar{s}) = \ell-1$ for all j and $\bar{s}$, the codimension of $EN(\bar{u})$ in $N(\bar{u})$--for generic $\bar{u}$--does not have a positive lower bound that is invariant of the choice of Φ, i.e., $EN(\bar{u})$ could be full-dimensional in $N(\bar{u})$ for some Φ. What is true (see Proposition 3.2) is that $N(\bar{u})$ has codimension $n(\ell-1)$ in S; and the set of efficient strategies $E(\bar{u})$ has codimension $(n-1)(\ell-1)$ in S. If the two sets $N(\bar{u})$ and $E(\bar{u})$ were to intersect transversally, then the codimension of $EN(\bar{u}) = N(\bar{u}) \cap E(\bar{u})$ would be $(n-1)(\ell-1)$ in $N(\bar{u})$. Such is indeed the generic picture if we vary not only the utilities $\bar{u}$ but <u>also</u> the mechanism Φ (remark 3.4). This picture, however, provides no information about what happens for an arbitrarily fixed Φ.

The analysis for fixed Φ hinges on a certain set S_{UO} and its image $\Phi(S_{UO})$. Let $V_j(\bar{s})$ be the normal direction to $Y_j(\bar{s})$ at the point $\Phi_j(\bar{s})$. Put $S_{UO} = \{\bar{s} \in S\colon\ Y_1(\bar{s}) = \ldots = Y_n(\bar{s})\}$. <u>Note that S_{UO} depends only on the mechanism</u> Φ. We show (Proposition 2.2) that--for <u>any</u> choice of utilities--an N.E. is efficient if, and only if, it lies in S_{UO}, i.e.,

$EN(\bar{u}) = N(\bar{u}) \cap S_{UO}$ for any $\bar{u}$. Therefore we nickname S_{UO} the set of "ultra-optimal" points of Φ. <u>The analysis of the efficiency of N.E. for any Φ is reduced to the analysis of its S_{UO}</u>. If the codimension of $\Phi(S_{UO})$ (in the space of reallocations) is n or more, then--for generic $\bar{u}$--the set $EN(\bar{u})$ is empty, i.e., all N.E. are inefficient. This is Proposition 3.1. In general we have a somewhat watered-down version of this result. Proposition 3.2 states that if S_{UO} has codimension t in S, then $EN(\bar{u})$ also has codimension t in $N(\bar{u})$ for generic $\bar{u}$. (Thus if $t > \dim N(\bar{u}) = \dim S - n(\ell-1)$, then again $EN(\bar{u})$ is empty... see Corollary 3.3.) Finally we sketch an argument (remark 3.4) that the codimension of S_{UO} in S is $(n-1)(\ell-1)$ for a generic choice of the mechanism Φ. Putting this together with Proposition 3.2 gives the overall picture of inefficiency of the N.E.

The role of S_{UO} is decisive for our analysis in the economically relevant case: $\dim Y_j(\bar{s}) = \ell-1$ for all j and all $\bar{s}$. The results here are also sharper. As noted earlier, for any Φ,

$$\text{codim } N(\bar{u}) \text{ in } S = n(\ell-1)$$

$$\text{codim } E(\bar{u}) \text{ in } S = (n-1)(\ell-1)$$

generically in $\bar{u}$ (Proposition 3.2). If $\dim S > n(\ell-1) + (n-1)(\ell-1)$ one might think that it is possible for $N(\bar{u})$ and $E(\bar{u})$ to have a transversal intersection. The points in this intersection (i.e., in $EN(\bar{u})$) would then constitute robust, efficient N.E.'s as we vary the utilities $\bar{u}$. But, in the analysis via S_{UO}, we can detect instances of Φ when $N(\bar{u})$ and $E(\bar{u})$ do not intersect for generic $\bar{u}$, even if $\dim S > n(\ell-1) + (n-1)(\ell-1)$.

This is the main thrust of Proposition 3.1 and Corollary 3.3. Moreover S_{UO} provides a simple method for checking inefficiency. If one is handed a Φ, one computes S_{UO} and $\Phi(S_{UO})$, and the dimensions of both sets. The inefficiency properties of N.E. for generic $\bar{u}$ can then be read off from this data.

In section 4 we apply our results to the "sell-all" and "buy-sell" mechanisms due to Shapley and Shubik. In the "buy-sell" mechanism $\dim S = 2n(\ell-1) > n(\ell-1) + (n-1)(\ell-1)$ and we are in the situation when robust, efficient N.E. cannot a priori be ruled out by a straight dimension count. But we find that, in both these mechanisms, S_{UO} <u>consists of strategies which leave each player with his initial endowment</u>. Proposition 3.1 then implies that all N.E. are inefficient for generic $\bar{u}$. However this can be seen directly as well. It is obvious that the gradients of the traders' utilities at the initial endowment will not, for generic $\bar{u}$, all point in the same direction. When this is so the initial endowment is not efficient (see the easy argument in the proof of Proposition 2.2). The inefficiency of N.E. now follows immediately from Proposition 2.2. Thus the simple, but at least to us surprising, structure of S_{UO} in the Shapley-Shubik models yields an elementary analysis of them. This result was obtained in [2] for the "sell-all" mechanism. We should point out that we derive it in a significantly simpler way. The existence of S_{UO} was not noticed in [2] necessitating a "brute force" argument. Here a brisk calculation leads to S_{UO} and thereby to the same result.

Two notions of efficiency ("economic"--which we have discussed so far--and "strategic," see section 2) are considered. The

conclusions above also carry over to strategic efficiency. Here it is no longer necessary to assume that the range of Φ is the space of reallocations. Depending again only on Φ, there is a set $ST_{UO} \subset S$ such that, for any choice of utilities, all strategically efficient N.E. lie in ST_{UO} (but now there may exist N.E. in ST_{UO} which are also strategically inefficient). ST_{UO} typically has positive codimension in S and, when this is the case, the set of strategically efficient N.E. has the same codimension in the set of all N.E.

Returning to the Shapley-Shubik mechanisms, it turns out that in both of them ST_{UO} consists of strategies that leave at least one player with his initial endowment. This leads to the strategic inefficiency of N.E.

Throughout we restrict ourselves to N.E. that do not occur on the boundary of the strategy-sets. This is largely a matter of technical convenience. If the strategy-sets have "nice" boundaries (e.g., are simplices) and Φ is defined smoothly in a neighborhood of S, our analysis can be carried through to the boundary. In fact $k_j = \dim Y_j(\bar{s})$ becomes variable with $\bar{s}$ now, and we need to "patch" together the cases for different k_j (remark 3.3).

2. Ultra-Optimal and Ultra-Inoptimal Points

We begin with the following set-up which is somewhat more general than the market mechanism described earlier. Assume that there are n players and that associated to each player j are:

(i) a strategy set S_j

(ii) an outcome space Y_j

(iii) a map $\Phi_j : S_1 \times \ldots \times S_n \to Y_j$.

Let $S = S_1 \times \ldots \times S_n$. A choice of strategies $\overline{s} = (s_1, \ldots, s_n) \in S$ determines the outcome $\Phi_j(\overline{s}) \in Y_j$ for the j^{th} player.

We assume throughout that Y_j is a smooth manifold and that the utility function u_j of the j^{th} player is a C^1-function on Y_j. A choice of utilities $\overline{u} = (u_1, \ldots, u_n)$ together with the maps Φ_j, defines the strategic market game. Let $Y = Y_1 \times \ldots \times Y_n$ and $\Phi : S \to Y$ be the map $\Phi_1 \times \ldots \times \Phi_n$.

Generally, if M is a manifold of dimension d, we will use the following notation:

a) $T_m(M)$ is the tangent space to M at m (it is a vector space of dimension d)

b) $T_m^*(M)$ is the cotangent space to M at m.

Recall that the cotangent space $T_m^*(M)$ is, by definition, the dual vector space to $T_m(M)$. For $v \in T_m(m)$ and $w \in T_m^*(M)$, we denote the value of w at v by $w \cdot v \in R$.

For this section, there is no need to impose any differentiable structure on the S_j or Φ_j. <u>We consider market mechanisms</u> Φ <u>which satisfy the following assumption</u> (see, however, remarks 3.3 and 3.5):

<u>Assumption 1</u>: If all but the j^{th} player fix their strategies, then the j^{th} player can, by changing his own strategy, span a submanifold of Y_j of codimension one. (A submanifold of codimension one is called a hypersurface.)

Let $\overline{s} = (s_1, \ldots, s_n) \in S$. Let $Y_j(\overline{s})$ denote the hypersurface in Y_j that the j^{th} player can span when the k^{th} player remains fixed

at s_k for all $k \neq j$. We will call $Y_j(\bar{s})$ the j^{th} player's holding hypersurface at $\bar{s}$. For $y \in Y_j(\bar{s})$, the tangent space $T_y(Y_j(\bar{s}))$ is a codimension one linear subspace of $T_y(Y_j)$. Set

$$V_j(\bar{s}) = \{w \in T^*_{\Phi_j(\bar{s})} : w \cdot v = 0 \quad \text{for all} \quad v \in T_{\Phi_j(\bar{s})}(Y_j(\bar{s}))\} .$$

Thus $V_j(\bar{s})$ is the one-dimensional space of linear functions on $T_{\Phi_j(\bar{s})}(Y_j(s))$ which vanish on $T_{\Phi_j(\bar{s})}(Y_j(\bar{s}))$.

Let ∇u_j be the gradient of u_j, that is, the vector of derivatives of u_j with respect to outcome variables on Y_j. For $y \in Y_j$, $\nabla u_j(y) \in T^*_y(Y_j)$ and $\nabla u_j(y) \cdot v$ is the directional derivative of u_j in the direction v.

A choice of strategies $\bar{s} = (s_1, \ldots, s_n) \in S$ is called a <u>Nash Equilibrium</u> for the utility functions $\bar{u} = (u_1, \ldots, u_n)$ if, for all j,

$$u_j(\Phi_j(s_1, \ldots, s_n)) \geq u_j(\Phi_j(s_1, \ldots, s_{j-1}, t, s_{j+1}, \ldots, s_n)) \text{ for all } t \in S_j .$$

Let $N(\bar{u})$ denote the subset of S of Nash equilibria for the utilities $\bar{u}$.

<u>Lemma 2.1</u>: Let $\bar{s} \in N(\bar{u})$. Then

$$\nabla u_j(\Phi_j(\bar{s})) \in V_j(\bar{s})$$

for all $j = 1, \ldots, n$.

<u>Proof</u>: Suppose that for some j and $\bar{s} \in S$, $\nabla u_j(\Phi_j(\bar{s})) \notin V_j(\bar{s})$. Then there is a vector $v \in T_{\Phi_j(\bar{s})}(Y_j(\bar{s}))$ such that $\nabla u_j(\Phi_j(\bar{s})) \cdot v > 0$, and v defines a direction in $Y_j(\bar{s})$ along which u_j is increasing. Player j can move in this direction by changing only his own strategy, hence $\bar{s} \notin N(\bar{u})$.

Remark 2.1: Suppose that the $Y_j(\bar{s})$ are contained in Euclidean space and the sets $\{x \in R^m : x \leq y$ for some y in $Y_j(\bar{s})\}$ are convex. Then if utilities are concave and non-decreasing in each variable, any local maximum on $Y_j(\bar{s})$ will be a global maximum and the condition of Lemma 2.1 in this case is also sufficient for $\bar{s}$ to be in $N(\bar{u})$.

Fix $Y_0 \subset Y_1$ such that $\Phi(S) \subset Y_0$. We shall regard Y_0 as the set of outcomes that are feasible from a purely economic point of view, though not necessarily achievable through a strategy-choice.

Definition: Given utilities $\bar{u} = (u_1, \ldots, u_n)$, a strategy choice $\bar{s} \in S$ is called:

a) economically efficient if there does not exist a $y = (y_1, \ldots, y_n) \in Y_0$ such that

$$u_j(y_j) \geq u_j(\Phi_j(\bar{s})) \quad \text{for all} \quad j = 1, \ldots, n$$

with strict inequality for some j.

b) strategically efficient if there does not exist a strategy choice $\bar{s}' \in S$ such that

$$u_j(\Phi_j(\bar{s}')) \geq u_j(\Phi_j(\bar{s})) \quad \text{for all} \quad j = 1, \ldots, n$$

with strict inequality for some j.

Definitions a) and b) differ only when the map $\Phi : S \to Y_0$ is not onto. Let $E(\bar{u})$ and $E_S(\bar{u})$ be the sets of economically efficient and stratetically efficient strategies respectively. Then $E(\bar{u}) \subset E_S(\bar{u})$. Set

$$EN(\bar{u}) = E(\bar{u}) \cap N(\bar{u})$$

$$EN_S(\bar{u}) = E_S(\bar{u}) \cap N(\bar{u}) .$$

For the rest of this section, we consider a market mechanism where the Y_j represent the spaces of final holdings of the players. Let $R^\ell_{++} = \{y = (y^1, \ldots, y^\ell) \in R^\ell : y^i > 0$ for all $i\}$ and assume that $Y_j = R^\ell_{++}$ for all j. For each player j, let $a_j = (a^1_j, \ldots, a^\ell_j) \in Y_j$ be the "initial endowment" of player j and set

$$Y_o = \{(y_1, \ldots, y_n) \in Y_1 \times \ldots \times Y_n : \sum_{j=1}^{n} y_j = \sum_{j=1}^{n} a_j\} .$$

Y_0 represents the space of reallocations of the initial endowment such that each player holds a positive amount in each commodity. Assume that Φ maps S to $Y_0 \subset Y_1 \times \ldots \times Y_n$, i.e.,

$$\sum_{j=1}^{n} \Phi_j(\bar{s}) = \sum_{j=1}^{n} a_j$$

for all $\bar{s} \in S$.

<u>Definition</u>: A strategy $\bar{s} \in S$ is called

a) <u>Ultra-optimal</u> if the one-dimensional subspaces $V_j(\bar{s})$ coincide for all j.

b) <u>Ultra-inoptimal</u> if, for some pair j and k, $V_j(\bar{s}) \neq V_k(\bar{s})$.

Let S_{UO} and S_{UI} denote the subsets of ultra-optimal and ultra-inoptimal points of S. It is clear that $S = S_{UO} \cup S_{UI}$ and $S_{UO} \cap S_{UI} = \emptyset$.

Let $\tilde{U}$ be the space of C^1-functions u on R^ℓ such that:

a) u is strictly concave: $u(tP + (1-t)Q) > tu(P) + (1-t)u(Q)$ for all $0 < t < 1$ and $P, Q \in R^\ell$.

b) $\nabla u(y) = (\partial u(y)/\partial x_1, \ldots, \partial u(y)/\partial x_\ell)$ is a vector with strictly positive components for all $y \in R^\ell$.

Proposition 2.2: Let $\Phi : S \to Y_0$ be a market mechanism satisfying Assumption 1. Then the decomposition $S = S_{UO} \cup S_{UI}$ has the following property: for all $\bar{u} = (u_1, \ldots, u_n) \in \tilde{U}^n$

$$EN(\bar{u}) = S_{UO} \cap N(\bar{u}) .$$

In other words, if $\bar{s} \in N(\bar{u}) \cap S_{UO}$, then $\bar{s}$ is economically efficient and if $\bar{s} \in N(\bar{u}) \cap S_{UI}$, then $\bar{s}$ is economically inefficient.

Proof: Let $\bar{s} \in N(\bar{u}) \cap S_{UO}$. By Lemma 1.1, $\nabla u_j(\Phi_j(\bar{s})) \in V_j(\bar{s})$ for all j and since the $V_j(\bar{s})$ all coincide, there is a single vector v with positive components such that

$$\nabla u_j(\Phi_j(\bar{s})) = \lambda_j v$$

for some positive number λ_j, for all j. If $\bar{s}$ were not economically efficient there would exist vectors $x_1, \ldots, x_n \in R^\ell$ such that $\sum_{j=1}^{n} x_j = 0$, $\Phi_j(\bar{s}) + x_j \in Y_j$, and such that

$$u_j(\Phi_j(\bar{s}) + x_j) \geq u_j(\Phi_j(\bar{s})) \text{ for all } j$$

with strict inequality for some j. Since the u_j are strictly concave

$$u_j(\Phi_j(\bar{s}) + tx_j) \geq u_j(\Phi_j(\bar{s}))$$

for all $0 < t < 1$ with strict inequality for some j. Hence $\nabla u_j(\Phi_j(\bar{s}))\cdot x_j \geq 0$ for all j with strict inequality for some j. Since $\nabla u_j(\Phi_j(\bar{s})) = \lambda_j v$, we have $v\cdot x_j \geq 0$ for all j with strict inequality for some j and this contradicts the assumption $\sum_{j=1}^{n} x_j = 0$.

Now suppose that $\bar{s} \in S_{UI} \cap N(\bar{u})$. Then there is a pair j and k such that $V_j(\bar{s}) \neq V_k(\bar{s})$, and since $\nabla u_j(\Phi_j(\bar{s})) \in V_j(\bar{s})$ and $\nabla u_k(\Phi_k(\bar{s})) \in V_k(\bar{s})$, there is a vector $x \in R^\ell$ such that

$$\nabla u_j(\Phi_j(\bar{s}))\cdot x > 0$$

$$\nabla u_k(\Phi_k(\bar{s}))\cdot(-x) > 0 \ .$$

Hence for t sufficiently small and positive, $\Phi_j(\bar{s}) + tx \in Y_j$, $\Phi_k(\bar{s}) - tx \in Y_k$, and:

$$u_j(\Phi_j(\bar{s}) + tx) > u_j(\Phi_j(\bar{s}))$$

$$u_k(\Phi_k(\bar{s}) + tx) > u_k(\Phi_k(\bar{s}))$$

and the reallocation assigning $u_i(\Phi_i(\bar{s}))$ to player i for $i \neq j$, k and $\Phi_j(\bar{s}) + tx$ (resp. $\Phi_k(\bar{s}) - tx$) to player j (resp. k) shows that $\bar{s}$ is not economically efficient.

3. Inefficiency of Nash Equilibria

In this section we retain the set-up of Proposition 2.2. The initial endowment vectors a_j and a market mechanism $\Phi : S \to Y_0$ are fixed, where Φ satisfies Assumption 1. We take the space of utility functions $\bar{U}$ to be as in section 1 except that now we must also require that all $u \in \bar{U}$ be C^{r+2}--functions where $r = \max\{\dim S - n(\ell-1), (n-1)\ell\}$. The topology on these functions will be as follows. Let Z be the cube $\{y \in R^\ell_+ : -\varepsilon < y^i < \sum_{j=1}^{n} a^i_j + \varepsilon\}$ in R^ℓ, where ε is any positive number. Then Z contains all possible final holdings of any trader. Let

$$\|u\| = \sup_D \sup_{y \in Z} |Du(y)|$$

where Du ranges over all derivatives of u of order 0, 1, ..., r+2. The norm $\| u \|$ makes $U = \{u \in \overline{U} : \| u \| \text{ is finite}\}$ into a Banach space--see Theorem 10.2 in [1].

Proposition 3.1. Suppose $\Phi(S_{U0})$ is contained in a submanifold M of Y_0, where codimension of $M \geq n$. Then there exists an open dense set U_0 of U^n such that, for all $\overline{u} = (u_1, \ldots, u_n)$ in U_0, $EN(\overline{u})$ is empty.

Proof: Let $W = \{z_1, \ldots, z_n = z_i \in Z, \sum_{i=1}^{n} z_i = \sum_{i=1}^{n} a_i\}$. Then W is relatively open in the affine span of Y_0 and contains the closure $\overline{Y}_0$ of Y_0 in its relative interior. Consider the map

$$\psi : U^n \times W \to W \times \text{Mat}\ (n,\ell)$$

given by

$$(\overline{u},\overline{y}) \quad (u_1, \ldots, u_n; y_1, \ldots, y_n) \to \left[y_1, \ldots, y_n, \begin{bmatrix} \nabla u_1(y_1) \\ \vdots \\ \nabla u_n(y_n) \end{bmatrix} \right]$$

where Mat (n,ℓ) is the set of $n \times \ell$ matrices with positive entries. For $A \in$ Mat (n,ℓ), the rows of A will be denoted by $A_1, \ldots, A_n$. Put

$$E = \{A \in \text{Mat}\ (n,\ell) : A_j = \lambda_j v \text{ for some } v \in R^{\ell}_{++} \text{ and } \lambda_j > 0;\ j = 1, \ldots, n\}$$

Then E is a submanifold of codimension $(n-1)(\ell-1)$ in Mat (n,ℓ).

It is clear that Φ is transverse to every submanifold of $W \times$ Mat (n,ℓ), in particular to $M \times E$. Let $\psi_{\overline{u}} : W \to W \times$ Mat (n,ℓ) be the map defined by $\psi_{\overline{u}}(\overline{y}) = \psi(\overline{u},\overline{y})$. By the transversal density and openness theorems (see [1]) there is an open dense set U_0 of U such that, if $\overline{u} \in U_0$, then $\psi_{\overline{u}}$ is transverse to $M \times E$ at every $y \in \overline{Y}_0$ (recall that

$\overline{Y}_0$ is the closure of Y_0 and is compact.) So, for $\overline{u} \varepsilon U_0$ (and viewing $\psi_{\overline{u}}$ as restricted to the domain $Y_0 \subset W$),

$$\begin{aligned} \operatorname{codim} \psi_{\overline{u}}^{-1}(M \times E) &= \operatorname{codim} M \times E \\ &> (n-1)(\ell-1) + n-1 \\ &= (n-1)\ell \\ &= \dim Y_0 . \end{aligned}$$

This implies that $\psi_{\overline{u}}^{-1}(M \times E)$ is empty if $\overline{u} \varepsilon U_0$.

But if $\overline{s} \varepsilon EN(\overline{u})$ and $\Phi(\overline{s}) = \overline{y}$, then $\psi_{\overline{u}}(\overline{y}) \varepsilon \Phi(S_{UO}) \times E \subset M \times E$ by Proposition 2.2 (and its proof), i.e., $\overline{y} \varepsilon \psi_{\overline{u}}^{-1}(M \times E)$. This shows that $EN(\overline{u})$ is empty if $\overline{u} \varepsilon U_0$.

Remark 3.1. Clearly Proposition 3.1 continues to hold if $\Phi(S_{UO})$ is contained in a finite union of submanifolds of Y_0, each of which has codimension at least n.

When $\Phi(S_{UO})$ has codimension n-1 or less, there may be robust, efficient N.E.'s. Nevertheless "most" N.E.'s are still inefficient as our next proposition shows. Now we assume, in addition, that the S_j are bounded, smooth manifolds in Euclidean spaces; for simplicity, suppose they are open sets. Each Φ_j is defined smoothly on an open set T which contains the closure $\overline{S}$ of S. We require Assumption 1 to hold for Φ_j on all of T. Also T is chosen suitably small so that $\Phi(T) \subset W$.

Proposition 3.2. There is an open dense set U^* of U^n such that for all $\overline{u} \varepsilon U^*$:

(a) $N(\bar{u})$ is contained in a submanifold of codimension $n(\ell-1)$ in S

(b) $E(\bar{u})$ is a submanifold of codimension $(n-1)(\ell-1)$ in S

(c) If S_{UO} is contained in a finite union of submanifolds of codimension at least t in S, then $EN(\bar{u})$ is contained in a finite union of submanifolds of codimension at least $n(\ell-1) + t$ in S.

<u>Proof</u>: Let ψ^* be the map

$$U^n \times T \to T \times \text{Mat}\ (n,\ell)$$

given by

$$(u_1, \ldots, u_n; \bar{s}) \to \begin{bmatrix} s, & \begin{matrix} \nabla u_1(\Phi(\bar{s})) \\ \vdots \\ \nabla u_n(\Phi(\bar{s})) \end{matrix} \end{bmatrix}$$

Let

$$N^* = \{(\bar{s},A) \in T \times \text{Mat}\ (n,\ell) : A_j \in V_j(\bar{s})\},$$

$E^* = \{(\bar{s},A) \in T \times \text{Mat}\ (n,\ell) : A_j = \lambda_j v$ for some $U \in R_{++}$ and $\lambda_j > 0,\ j = 1, \ldots,$

Then N is a submanifold of $T \times \text{Mat}\ (n,\ell)$ of codimension $n(\ell-1)$, and E is of codimension $(n-1)(\ell-1)$. It is clear that ψ^* is transverse to every submanifold of $T \times \text{Mat}\ (n,\ell)$. Therefore there is an open dense set U^* in U^n such that, if $\bar{u} \in U^*$, $\psi^*_{\bar{u}}$ is transverse to N^* and to E^* at each $\bar{s}$ in the compact set $\bar{S}$. Since $N(\bar{u}) \subset \psi^{*-1}_{\bar{u}}(N^*)$ and $E(\bar{u}) = \psi^{*-1}_{\bar{u}}(E^*)$ conclusions (a) and (b) follow. If $S_{UO} \subset M_1 \cup \ldots \cup M_r$ where the M_j are submanifolds of S of codimension at least t, then by Proposition 2.2,

$$EN(\overline{u}) \subset \bigcup_{j=1}^{r} \psi^{*}_{\overline{u}}{}^{-1}(M_j \times N^*)$$

If U^* is chosen so that $\psi^*_{\overline{u}}$ is also transverse to each of $M_j \times N^*$, then conclusion (c) also follows.

Corollary 3.3. If codim $S_{UO} >$ dim $S - n(\ell-1)$, then $EN(\overline{u})$ is empty for generic $\overline{u}$.

Remark 3.2. On the face of it, Proposition 3.2 leaves open the possibility that all N.E. may be economically efficient. But a little reflection shows that this is not so. Suppose Φ satisfies the condition of remark 2.1. Then

$$N(\overline{u}) = \{\overline{s} \in S : \psi(\overline{s},\overline{u}) \in N\} .$$

From the proof of Proposition 3.2 we see that, for generic $\overline{u} \in U^n$,

(a) $N(\overline{u})$ is either empty or a manifold of codimension $n(\ell-1)$ in S,

(b) $EN(\overline{u})$ is either empty or contained in a finite union of submanifolds with codimension at least t in the Nash manifold $N(\overline{u})$.

Remark 3.3. Suppose dim $Y_j(\overline{s})$ = codim $V_j(\overline{s}) = k_j \leq \ell-1$ for $j = 1, \ldots, n$ and $\overline{s} \in S$, with strict inequality for at least one j. Let N^* be as before, i.e.,

$$N^* = \{(\overline{s},A) \in S \times \text{Mat}\ (n,\ell) : A_j \in V_j(\overline{s}) \text{ for } j = 1, \ldots, n\} .$$

Put $k = k_1 + \ldots + k_n$. Then codimension $N^* = k$, so by the same proof as that of Proposition 3.1, we get--for generic $\overline{u}$--

(a)* $N(\overline{u})$ is contained in a submanifold of codimension k in S.

Next define

$$EN^* = \{(\bar{s},A) \in S \times \text{Mat}\,(n,\ell) : A_j \in V_j(\bar{s}) \text{ and } A_j = \lambda_j v \text{ for some } v \in R^\ell$$
$$\text{and } \lambda_j > 0,\ j = 1, \ldots, n\}\ .$$

Put $k_* = \max_{1 \leq j \leq n} k_j$. Then EN^* is contained in a finite union of submanifolds of $S \times \text{Mat}\,(n,\ell)$, each of which has codimension at least $k_* + (n-1)(\ell-1)$. (To see this, suppose w.l.o.g. that $k_1 = \max_j k_j$. Then EN^* is contained, for instance, in:

$$\{(\bar{s},A) \in S \times \text{Mat}\,(n,\ell) : A_1 \in V_1(\bar{s}) \text{ and } A_j = \lambda_j A_1 \text{ for } \lambda_j > 0,\ j = 2, \ldots, n)\}\ .)$$

But

$$EN(\bar{u}) \subset \{\bar{s} \in S : \psi^*(\bar{s},\bar{u}) \in EN^*\}\ .$$

As shown in the proof of Proposition 3.2, the map

$$\psi_{\bar{u}} : S \to S \times \text{Mat}\,(n,\ell)$$

given by $\psi_{\bar{u}}(\bar{s}) = \psi(\bar{s},\bar{u})$ is transverse to every submanifold of its image for generic $\bar{u}$ in U^n. Thus, for generic $\bar{u}$, we have

> (b)* $EN(\bar{u})$ is contained in a finite union of submanifolds, each of which has codimension at least $k_* + (n-1)(\ell-1)$ in S.

Putting together (a)* and (b)* with Remark 3.1, the picture (for generic $\bar{u}$ in U^n) is: (i) $N(\bar{u})$ is either empty or a submanifold of S with codimension k, (ii) $EN(\bar{u})$ is either empty or contained in a finite union of submanifolds with codimension at least $k_* + (n-1)(\ell-1) - k > 0$ in $N(\bar{u})$.

This leaves the case of $k_j(\bar{s})$ varying with $\bar{s}$. If S and Φ are "nice" enough, we can suppose that there is a finite partition of S into manifolds $A_1, \ldots, A_\ell$ such that $k_1(\bar{s}), \ldots, k_n(\bar{s})$ are all constant on each A_i. Our analysis shows inefficiency of the N.E. for $\bar{s} \varepsilon A_i$ and generic $\bar{u}$. Putting together the ℓ cases, we get the inefficiency on S. This case arises naturally when traders are on the boundary of their strategy sets. To be precise suppose each $S_j = M_j^1 \cup \ldots \cup M_j^{k(j)}$ is a finite union of manifolds (e.g., S_j is a simplex) and that Φ is defined smoothly on a neighborhood of S. Take the A_i to be all possible products $M_1^{\alpha(1)} \times \ldots \times M_n^{\alpha(n)}$, $1 \leq \alpha(i) \leq k(i)$. Then, for "well-behaved" Φ (e.g., the Shapley-Shubik mechanisms), it will turn out that $k_1(\bar{s}), \ldots, k_n(\bar{s})$ are constant for $\bar{s} \varepsilon A_i$. When this happens our analysis obviously extends to the boundary. See [3] for an explicit treatment of the boundary for similar questions in a purely game-theoretic context.

<u>Remark 3.4</u>. For $\bar{s} \varepsilon S$, the condition that all $V_j(\bar{s})$ coincide is defined "in general" by $(n-1)(\ell-1)$ equations. It seems likely that if the S_j are bounded open subsets of Euclidean space, then a rigorous argument could be given to show that the codimension of S_{UO} in S in $(n-1)(\ell-1)$ for a generic class of market mechanisms. We indicate briefly how this might be done. Let Ω be the set of smooth maps Φ from S to Y_0 which satisfy Assumption 1. Then $\Phi = \Phi_1 \times \ldots \times \Phi_n$ where $\Phi_j : S \to Y_j$ and for each $\bar{s} \varepsilon S$, $V_j(\bar{s})$ defines a line in R^ℓ. Let $P^{\ell-1}$ denote the projective space of all lines in R^ℓ. It is a compact manifold of dimension $(\ell-1)$ whose points correspond to lines in R^ℓ. Let ψ be the map

$$\psi : \Omega \times S \to S \times (P^{\ell-1})^n$$

$$(\Phi,\bar{s}) \to (s,V_1(\bar{s}), \ldots, V_n(\bar{s})) .$$

and let $UO = \{(\bar{s},V_1, \ldots, V_n) \varepsilon S \times (P^{\ell-1})^n : V_1 = V_2 = \ldots = V_n\}$. Then for all $\Phi \varepsilon \Omega$,

$$S_{UO}(\Phi) = \{\bar{s} \varepsilon S : \psi(\Phi,\bar{s}) \varepsilon UO\}$$

where $S_{UO}(\Phi)$ denotes the set of ultra-optimal points in S with respect to the market mechanism Φ. It can probably be shown that, with respect to suitable topologies, there is a dense open set $\Omega_0 \subset \Omega$ such that Ω_0 is a Banach manifold. One would then show that ψ restricted to $\Omega_0 \times S$ is transverse to the submanifold UO of $S \times (P^{\ell-1})^n$. Since the codimension of UO in $S \times (P^{\ell-1})^n$ is $(n-1)(\ell-1)$, the desired conclusion would follow from the transversal density and openness theorems.

Remark 3.5. A "local" version of Assumption 1 would have been sufficient for our analysis. For any $\bar{s} = (s_1, \ldots, s_n)$ and $1 \leq j \leq n$ suppose:

(a) There a neighborhood of $\Phi_j(\bar{s})$ in $Y_j(\bar{s})$ which is diffeomorphic to $R^{\ell-1}$

or

(b) There is a neighborhood N of s_j in S_j such that $\Phi_j(s_1, \ldots, s_{j-1}, t, s_{j+1}, \ldots, s_n) = t\varepsilon N\}$ is diffeomorphic to $R^{\ell-1}$.

In either case, define $V_j(\bar{s})$ to be the appropriate normal and the analysis goes through exactly as before. The stronger Assumption 1 was made for ease of presentation.

4. The Shapley-Shubik Mechanisms ([8], [9], [10])

We now examine the results of Sections 1 and 2 in a special case: the Shapley-Shubik "buy-sell" mechanism. There are n players and ℓ commodities, where the ℓ^{th} commodity is treated as money. The initial endowment vectors $a_j = (a_j^1, \ldots, a^\ell) \in Y_j = R_{++}^\ell$ are given and the j^{th} player's strategies consist of a bid b_j^i of money to purchase commodity i and an offer to sell a quantity q_j^i of commodity i $(1 \leq i \leq \ell-1)$. Player j's strategy is represented by two vectors

$$b_j = (b_j^1, \ldots, b_j^{\ell-1}) \ , \ q_j = (q_j^1, \ldots, q_j^{\ell-1})$$

and

$$S_j = \{(b_1, q_j) \in R_{++}^{\ell-1} \times R_{++}^{\ell-1} : q_j^i < a_j^i \text{ and } \sum_{i=1}^{\ell-1} b_j^i < a_j\} \ .$$

S_j is an open set in $R^{2(\ell-1)}$ and dim S = $2n(\ell-1)$. Let

$$B^i = \sum_{j=1}^n b_j^i \ , \ Q^i = \sum_{j=1}^n q_j^i$$

be the total amounts bid and offered on commodity i. Then

$$p^i = \frac{B^i}{Q^i}$$

is the price formed on commodity i.

The outcome is given by distributing the total amount of each commodity offered among the players in proportion to their bids and the total amount of money bid on each commodity in proportion to the offers. The outcome of player j will be denoted by $x_j = (x_j^1, \ldots, x_j^\ell) \subset Y_j$ and is given by:

$$x_j^i = a_j^i - q_j^i + \frac{b_j^i}{p^i} \ (1 \leq i \leq \ell-1)$$

$$x_j^\ell = a_j^\ell - \sum_{i=1}^{\ell-1} b_j^i + \sum_{i=1}^{\ell-1} p^i q_j^i \ .$$

We compute the hypersurfaces $Y_j(\bar{s})$ and the lines $V_j(\bar{s})$ (or equivalently the normals to the holding hypersurfaces $Y_j(\bar{s})$) in the next lemma. Set

$$B_j^i = \sum_{k \neq j} b_k^i \ , \ Q_j^i = \sum_{k \neq j} q_k^i \ , \ p_j^i = \frac{B_j^i}{Q_j^i} \ .$$

Then p_j^i is the price on commodity i formed by the players other than j.

<u>Lemma 4.1</u>: a) The hypersurface $Y_j(\bar{s})$ is defined by the equation

$$x_j^\ell = a_j^\ell + \sum_{i=1}^{\ell-1} \frac{B_j^i(a_j^i - x_j^i)}{Q_j^i + a_j^i - x_j^i} \ .$$

b) A vector normal to $Y_j(\bar{s})$ at $\Phi_j(\bar{s})$ is given by:

$$((p^1)^2 \ p_j^1, \ (p^2)^2 \ p_j^2, \ \ldots, \ (p^{\ell-1})^2 \ p_j^{\ell-1}, \ 1) \ .$$

<u>Proof</u>: To prove a), we have to show that for $i = 1, \ldots, \ell-1$,

$$p^i q_j^i - b_j^i = \frac{B_j^i(a_j^i - x_j^i)}{Q_j^i + a_j^i - x_j^i}$$

and then summing over i gives the result. Since $p^i = B^i/Q^i$, $B_j^i = B^i - b_j^i$, and $Q_j^i = Q^i - q_j^i$, we have to check that

$$\frac{B^i}{Q^i} q_j^i - b_j^i = \frac{(B^i - b_j^i)(a_j^i - x_j^i)}{(Q^i - q_j^i + a_j^i - x_j^i)} \ .$$

Since

$$a_j^i - x_j^i = q_j^i - \frac{b_j^i}{p^i} = q_j^i - \frac{b_j^i Q^i}{B^i},$$ this is easily verified .

To prove b), note that B_j^i and Q_j^i do not depend on player j and a normal to a hypersurface in parametric form, as in a), is given by $(\partial x_j^\ell/\partial x_j^1, \ldots, \partial x_j^\ell/\partial x_j^{\ell-1}, -1)$ where x_j^ℓ is a function of $x_j^1, \ldots, x_j^{\ell-1}$ as in a). From a), we have

$$\frac{\partial x_j^\ell}{\partial x_j^i} = \frac{-B_j^i Q_j^i}{(Q_j^i + a_j^i - x_j^i)^2}$$

and it is easy to check that $Q_j^i + a_j^i - x_j^i = B_j^i/p^i$. Hence a normal is given by $((p^1)^2\ p_j^1, \ldots, (p^{\ell-1})^2\ p_j^{\ell-1}, 1)$.

A strategy is ultra-optimal when the lines spanned by the normals to the hypersurfaces $Y_j(\bar{s})$ coincide. Since the prices p^i coincide for all players, $\bar{s} \in S_{UO}$ if and only if the quantities p_j^i are independent of j for $i = 1, \ldots, \ell-1$. This gives the $(n-1)(\ell-1)$ independent equations defining S_{UO}.

Suppose that p_j^i is independent of j for all i, say $p_j^i = \lambda_i$. Then we have

$$\frac{B_i - b_j^i}{Q^i - q_j^i} = \lambda_i \text{ , or}$$

$$B^i - b_j^i = \lambda_i(Q^i - q_j^i) \tag{*}$$

for all j. Summing (*) over j shows that $\lambda_i = p^i$. Substituting this back in (*) gives:

$$p_j^i = \frac{b_j^i}{q_j^i} = p^i \text{ .}$$

When this holds, the outcome to each player is simply his initial endowment. We have:

Proposition 4.2. In the buy-sell mechanism, the set S_{UO} of ultra-optimal strategies consists of those strategies such that each player's outcome is his initial endowment, that is, such that:

$$b_j^i = p^i q_j^i \quad \text{for all } i \text{ and } j .$$

A variant of the buy-sell mechanism is the sell-all mechanism, in which all players are required to offer their entire endowment for sale, i.e., $q_j^i = a_j^i$ for all i and j. Clearly here again we have:

Proposition 4.3. In the sell-all mechanism, S_{UO} consists of the strategies such that $b_j^i = a_j^i p^i$ for all i and j and each such strategy gives each player his initial endowment as the outcome.

It turns out that the market mechanism in both cases blows at strategies at which the total bid or offer (or both) is zero for any commodity. However if we confine ourselves to the subset V of U given by $V = \{u \in U : a < \nabla u < b\}$, for some positive vectors a and b, then there exist positive numbers c and d such that (in either model)

$$N(\bar{u}) \subset T = \{\bar{s} \in S : c < p^i(\bar{s}) < d \quad \text{for} \quad i = 1, \ldots, \ell-1\}$$

for $\bar{u} \in V^n$. This is shown in Lemma 1 of [1] for the sell-all mechanism and can be shown for the buy-sell mechansim in the same way. By applying Proposition 3.1 to the set T for all $d > c > 0$, we obtain the next proposition.

Proposition 4.4. There is an open dense set $U_0 \subset U^n$ such that, for all $\bar{u} = (u_1, \ldots, u_n) \in U_0$, $EN(\bar{u})$ is empty in both the "sell-all" and the "buy-sell" mechanisms.

But, as we observed in the introduction, Proposition 4.4 can be seen directly from Proposition 2.2 without using Proposition 3.1.

5. Strategic Efficiency

Assume now that strategy sets S_j, outcome spaces Y_j, and maps $\Phi_j : S \to Y_j$ satisfying Assumption 1 are given, and assume that the S_j are smooth manifolds and that the Φ_j are smooth maps. To examine strategic efficiency, we will define a decomposition $S = ST_{UI} \cup ST_{UO}$ similar to the decomposition $S = S_{UI} \cup S_{UO}$ defined in Section 1 when Y_0 is a space of reallocations.

Let $d\Phi_j$ be the Jacobian of the map Φ_j. We may write

$$d\Phi_j = [d\Phi_{j1} \cdots d\Phi_{jn}]$$

where $d\Phi_{ji}$ is the matrix of partial derivatives of the j^{th} player's outcomes with respect to the i^{th} player's strategies. Assumption 1 of Section 2 will now be changed to:

<u>Assumption 1'</u>: For all j and all $\bar{s} \in S$, $d\Phi_{jj}(\bar{s})$ has rank equal to $(\dim Y_j - 1)$.

This is the infinitesimal version of Assumption 1.

Let $\nabla_s u_j$ be the gradient of partial derivatives of u_j with respect to the strategic variables. By the chain rule:

$$\nabla_s u_j = (\nabla u_j) \cdot d\Phi_j \ .$$

According to a simple lemma of Smale, if $\bar{s} \in E_S(\bar{u})$, then the vectors $\nabla_s u_j(\bar{s})$ are linearly dependent. Let $v_j(\bar{s})$ denote a non-zero vector in $V_j(\bar{s})$; it is determined up to scalar multiples. Smale's lemma and

Lemma 2.1 yield the following. (Concavity of utilities is not needed here, only that the gradients be nowhere-vanishing.)

Lemma 5.1: Let ST_{UO} be the set of $\bar{s} \in S$ with the property: the vectors $(v_j(\bar{s}))\cdot d\Phi_j$ are linearly dependent. Then for all utilities $\bar{u} = (u_1, \ldots, u_n)$:

$$N(\bar{u}) \cap E_S(\bar{u}) \subset ST_{UO} .$$

If we let ST_{UI} be the complement of ST_{UO} in S, then $S = ST_{UI} \cup ST_{UO}$. For all choices of utilities (with nowhere-vanishing gradients), the efficient Nash equilibria, if there are any, all lie in ST_{UO}. An analogue of Proposition 3.2 is also true in the context of stratetic efficiency, where S_{UO} is replaced by ST_{UO}. Generically, the set of efficient Nash equilibria will have codimension t in $N(\bar{u})$ if ST_{UO} has codimension t in S. Let m = dim S. Since $m \geq n$, the condition that the n vectors $v_j(\bar{s})\cdot d\Phi_j$ be linearly dependent is defined by $\binom{m}{n}$ equations--those obtained by setting the determinants of all $n \times n$ minors of the matrix with rows $v_j(\bar{s})\cdot d\Phi_j$ $(j = 1, \ldots, n)$ equal to zero. These equations may not define independent conditions on $\bar{s}$, but one may expect that for a generic class of Φ, ST_{UO} has positive codimension in S. This is true for the Shapley-Shubik models, as we shall see below, and this explains the inefficiency result of [2], according to which, generically in utilities, the Nash equilibria are strategically inefficient in the sell-all model.

Consider the buy-sell model with n players and ℓ commodities (the ℓ^{th} commodity is money). With notation as before, let:

$$C^i_{jk} = \begin{bmatrix} \dfrac{\partial x^1_j}{\partial b^i_k} \\ \vdots \\ \dfrac{\partial x^\ell_j}{\partial b^i_k} \end{bmatrix} , \quad B^i_{jk} = \begin{bmatrix} \dfrac{\partial x^1_j}{\partial q^i_k} \\ \vdots \\ \dfrac{\partial x^\ell_j}{\partial q^i_k} \end{bmatrix}$$

so that the $\ell \times 2(\ell-1)$ matrix $d\Phi_{jk}$ is given by

$$d\Phi_{jk} = [C^1_{jk} \dots C^{\ell-1}_{jk}, B^1_{jk} \dots B^{\ell-1}_{jk}] .$$

It is easy to compute that

$$C^i_{jj} = \left[0, \dots, 0 , \underbrace{\frac{B^i - b^i_j}{B^i p^i}}_{i^{th} \text{ place}} , \dots, 0 , \frac{q^i_j - Q^i}{Q^i}\right]$$

$$C^i_{jk} = \left[0, \dots, 0 , \underbrace{\frac{-b^i_j}{B^i p^i}}_{i^{th} \text{ place}} , \dots, 0 , \frac{q^i_j}{Q^i}\right] \quad \text{for} \quad j \neq k$$

$$B^i_{jj} = \left[0, \dots, 0 , \underbrace{\frac{b^i_j - B^i}{B^i}}_{i^{th} \text{ place}} , \dots, 0 , p^i\left(\frac{Q^i - q^i_j}{Q^i}\right)\right]$$

$$B^i_{jk} = \left[0, \dots, 0 , \underbrace{\frac{b^i_j}{B^i}}_{i^{th} \text{ place}} , \dots, 0 , -p^i\left(\frac{q^i_j}{Q^i}\right)\right] \quad \text{for} \quad j \neq k .$$

As we saw in Section 4,

$$v_j(\bar{s}) = \left[(p^1)^2 \left(\frac{q_j^1 - Q^1}{B^1 - b_j^1} \right), \ldots, (p^{\ell-1})^2 \left(\frac{q_j^{\ell-1} - Q^{\ell-1}}{B^{\ell-1} - b_j^{\ell-1}} \right), -1 \right]$$

defines a vector in $V_j(\bar{s})$. For $j \neq k$, set

$$w_j = \left[\frac{b_j^1 - p^1 q_j^1}{B^1 - b_j^1}, \ldots, \frac{b_j^{\ell-1} - p^{\ell-1} q_j^{\ell-1}}{B^{\ell-1} - b_j^{\ell-1}}, p^1 \left(\frac{p^1 q_j^1 - b_j^1}{B^1 - b_j^1} \right), \ldots, p^{\ell-1} \left(\frac{p^{\ell-1} q_j^{\ell-1} - b_j^{\ell-1}}{B^{\ell-1} - b_j^{\ell-1}} \right. \right.$$

A short calculation shows that in the $n \times 2n(\ell-1)$ matrix

$$\begin{bmatrix} V_1(\bar{s}) \\ \vdots \\ V_n(\bar{s}) \end{bmatrix} \cdot d\Phi(\bar{s}) \text{ is equal to}$$

$$\left[\begin{array}{c|c|c|c|c} 0 \ldots 0 & w_1 & w_1 & & w_1 \\ w_2 & 0 \ldots 0 & w_2 & & w_2 \\ w_3 & w_3 & 0 \ldots 0 & \ldots & w_3 \\ \vdots & \vdots & \vdots & & \vdots \\ w_n & w_n & w_n & & 0 \ldots 0 \end{array} \right]$$
$$\underbrace{\qquad\qquad}_{2(\ell-1)}$$

In order for $\bar{s}$ to lie in ST_{UO}, the above matrix must have linearly dependent rows. It is easily checked that this is possible only if $w_j = (0, 0, \ldots, 0)$ for some j, in other words, only if $b_j^i = p^i q_j^i$ for $i = 1, \ldots, \ell-1$, for at least one player j. This gives:

Proposition 5.2. In the Shapley-Shubik buy-sell and sell-all mechanisms, ST_{UO} consists of those strategies such that the outcome for at least one player is his initial endowment, that is, for at least one j,

$$b_j^i = p^i q_j^i \quad \text{for} \quad i = 1, \ldots, \ell-1 .$$

The codimension of ST_{UO} is $(\ell-1)$.

This result and Proposition 4.2 show clearly the difference between economic and strategic efficiency of Nash equilibria. One can easily check that the holding hypersurfaces $Y_j(\bar{s})$ are concave in the Shapley-Shubik mechanisms. Hence, using Remark 3.1 and recalling that codim $N(\bar{u}) = n(\ell-1)$, we get:

Proposition 5.3. For generic $\bar{u}$, the set of strategically efficient N.E. is (i) either empty or a union of manifolds of codimension $\ell-1$ in $N(\bar{u})$ in the buy-sell mechanism, (ii) empty in the sell-all mechanism.

Remark 5.1. See [8] for explicit examples (with $n = 2$ and $\ell = 2$) which presage, as well as corroborate, our analysis of the buy-sell and sell-all mechanisms.

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